Then,

$$k = \frac{xy}{2}. \tag{1}$$

Now, $P = AC + CB + AB$, and for a minimum $dP/dy = 0$ and $\delta P/\delta\alpha = 0$. But, $AC = y/\sin\alpha = y \csc\alpha$; and since $\overline{CB}^2 = \overline{CD}^2 + \overline{DB}^2$ and $DB = AB - AD = x - y\cot\alpha$,

$$CB = [y^2 + (x - y\cot\alpha)^2]^{\frac{1}{2}} = \left(y^2 \csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}}.$$

Then

$$P = y\csc\alpha + \left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}} + \frac{2k}{y}.$$

$$\therefore\ \frac{dP}{dy} = \csc\alpha + \frac{1}{2}\left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{-\frac{1}{2}}\left(2y\csc^2\alpha - \frac{8k^2}{y^3}\right) - \frac{2k}{y^2} = 0,$$

which can be written

$$(y^2\csc\alpha - 2k)[(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)[^{\frac{1}{2}} - (2k + y^2\csc\alpha) = 0.$$

Hence,

$$y^2\csc\alpha - 2k = 0 \tag{2}$$

and

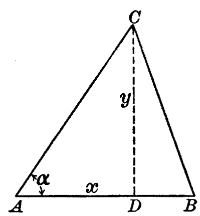
$$y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha = (2k + y^2\csc\alpha)^2,$$

which reduces to

$$-4ky^2\cot\alpha = 4ky^2\csc\alpha.$$

Hence,

$$-\cot\alpha = \csc\alpha \qquad \text{or} \qquad \alpha = 180°.$$




Substituting (1) in (2) we have $y^2\csc\alpha = xy$ or $\sin\alpha = y/x$; and since $\sin\alpha = y/AC$, $AC = x$, that is, $AC = AB$. Also

$$dP/d\alpha = -y\csc\alpha\cot\alpha + \tfrac{1}{2}(y^2\csc\alpha + 4k^2/y^2 - 4k\cot\alpha)^{-\frac{1}{2}}(-2y^2\csc^2\alpha\cot\alpha + 4k\csc^2\alpha) = 0;$$

or reducing,

$$\cos\alpha\cdot(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)^{\frac{1}{2}} = 2k - y^2\cot\alpha.$$

Squaring both sides and reducing, we get

$$(y^2\cot\alpha - k)(1 - \cos^2\alpha) = 0.$$

Hence,

$$1 - \cos^2\alpha = 0 \qquad \text{and} \qquad y^2\cot\alpha - k = 0. \tag{3}$$

Solving (2) and (3) for $\alpha$, we get $\tan\alpha = 2\sin\alpha$. Hence $\alpha = 60°$; and since we have shown above that $AB = AC$ and $\alpha = 60°$ is the included angle between these two lines, it follows that $CB = AB = AC$.

Also solved by FRANK IRWIN.

**391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.**

If $0 < \lambda < 1$ and $0 < x < \pi$ show that the function $(\sin\lambda x)/(\sin x)$ increases as $x$ increases.

## I. SOLUTION BY H. S. UHLER, Yale University.

Applying the well-known formula

$$\sin\theta = \theta\left[1 - \left(\frac{\theta}{\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{2\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{3\pi}\right)^2\right]\cdots, \qquad [\theta^2 < \infty],$$

to the problem under consideration we get

$$\frac{\sin \lambda x}{\sin x} = \frac{\lambda\left[1-\left(\frac{\lambda x}{\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{2\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{3\pi}\right)^2\right]\cdots}{\left[1-\left(\frac{x}{\pi}\right)^2\right]\left[1-\left(\frac{x}{2\pi}\right)^2\right]\left[1-\left(\frac{x}{3\pi}\right)^2\right]\cdots}.$$

Consider now the ratio of the $n$th binomial factor of the numerator to the $n$th or corresponding factor of the denominator:

$$\frac{1-\left(\frac{\lambda x}{n\pi}\right)^2}{1-\left(\frac{x}{n\pi}\right)^2} = \frac{n^2\pi^2 - \lambda^2 x^2}{n^2\pi^2 - x^2} = 1 + \frac{(1-\lambda^2)x^2}{n^2\pi^2 - x^2}.$$

For any chosen value of $n$ $(1, 2, 3, \cdots)$ the denominator of the last fraction decreases while the numerator increases as $x$ grows larger. The fraction is always finite and positive because of the hypotheses $0 < \lambda < 1$ and $0 < x < \pi$. Consequently, the ratio of any binomial in the numerator of the expression for $(\sin \lambda x)/(\sin x)$ to the corresponding binomial in the denominator increases with $x$. It is accordingly manifest that the product of an infinite number of such converging ratios increases as $x$ increases, and so the problem is solved.

## II. Solution by Frank Irwin, University of California.

We shall show that, for the values of $x$ in question, the derivative of the function is positive. This derivative is

$$\frac{\lambda\cdot\cos\lambda x\cdot\sin x - \sin\lambda x\cdot\cos x}{\sin^2 x}$$

or

$$\frac{1}{x\cdot\sin^3 x\cdot\sin\lambda x}(\lambda x\cdot\cot\lambda x - x\cdot\cot x).$$

This will be positive if $\lambda x\cdot\cot\lambda x > x\cdot\cot x$, that is, if $y\cdot\cot y$, let us say, continually decreases as $y$ increases from 0 to $\pi$. This is so, since its derivative, $\cot y - y\csc^2 y$, or $(\sin 2y - 2y)/2\sin^2 y$, is always negative.

A solution similar to the second was received from Elijah Swift.

**392. Proposed by HORACE OLSON, Student at The University of Chicago.**

Two right cylinders of radii $a$ and $b$, respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

## Solution by Elijah Swift, University of Vermont.

Assume $a > b$. Let the axis of the smaller cylinder be the $z$-axis, that of the larger, the $y$-axis. We see that the volume is

$$V = 8\int_0^b\int_0^{\sqrt{b-x^2}}\sqrt{a^2-x^2}\cdot dy\cdot dx = 8\int_0^b\sqrt{(a^2-x^2)(b^2-x^2)}\cdot dx.$$

This latter integral is elliptic, and may be expressed in terms of complete elliptic integrals of the first and second kinds. Letting $x = b\,\mathrm{sn}\,(y, b/a)$, the integral becomes

$$8ab^2\int_0^K\left\{1 - \frac{a^2+b^2}{a^2}\mathrm{sn}^2\,y + \frac{b^2}{a^2}\mathrm{sn}^4\,y\right\}dy.$$

This, in turn, may be integrated by reduction formulas and gives finally for the volume,

$$V = \tfrac{8}{3}a\left[(b^2-a^2)K\left(\frac{b}{a}\right) + (a^2+b^2)E\left(\frac{\pi}{2}, \frac{b}{a}\right)\right],$$

where $K$ denotes the complete elliptic integral of the first kind, $E$ the elliptic integral of the second kind.